Global generation of sheaves (Har II 5)

Let X be a scheme, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules.

Def: $\mathcal{F}$ is generated by global sections, or globally generated, if there is a set of global sections

$$\{s_i\} \subseteq \Gamma(X, \mathcal{F})$$

such that for each $x \in X$, the images of the s_i in $\mathcal{F}_x$ generate the stalk as an $\mathcal{O}_x$-module.

Note: If $s \in \Gamma(X, \mathcal{F})$ any section, we get a morphism of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \longrightarrow \mathcal{F}$$

By defining on each open set U

$$\begin{aligned} \mathcal{O}_X(U) &\longrightarrow \mathcal{F}(U) \\ 1 &\longmapsto s|_U \end{aligned}$$

So if $\mathcal{F}$ is globally generated, we sum the morphisms for each s_i and get

$$\bigoplus_i \mathcal{O}_X \longrightarrow \mathcal{F},$$

which is surjective on stalks and thus surjective.

Conversely, given a surjective map from a free sheaf, we get a set of sections of $\mathcal{F}$ that generate each stalk, so $\mathcal{F}$ is globally generated.

Ex: On $X = \operatorname{Spec} A$, every quasi-coherent sheaf $\mathcal{F}$ is globally generated, since $\mathcal{F} = \tilde{M}$, and $\tilde{M}_p = M_p$.

Ex: If S is graded and $X = \operatorname{Proj} S$, $\Gamma(X, \mathcal{O}(n)) = 0$ for $n < 0$, so $\mathcal{O}(n)$ is not globally generated in this case.

A theorem of Serre says that if $\mathcal{F}$ is coherent on "nice" projective schemes, a high enough twist will be globally generated. First we prove this in the case of projective space.

Lemma: Let $X = \mathbb{P}^r_A = \operatorname{Proj} A[x_0, \dots, x_r]$, A Noetherian, $\mathcal{F}$ coherent. Then there's some n_0 s.t. for $n \geq n_0$, $\mathcal{F}(n)$ is globally generated by finitely many global sections.

Pf: First cover X w/ $D_+(x_i)$. Since $\mathcal{F}$ is coherent, for each i, there's some f.g. module M_i over $B_i = A[x_0/x_i, \dots, x_n/x_i]$ s.t.

$$\mathcal{F}|_{D_+(x_i)} = \tilde{M}_i.$$

For each i, let $\{s_{i1}, s_{i2}, \dots\}$ be a finite generating set for M_i.

By an earlier lemma (from last section), there's some n s.t. $x_i^n s_{ij}$ extends to a global section t_{ij} of $\mathcal{F}(n)$. (Here we're applying lemma to $\mathcal{L} = \mathcal{O}_X(1)$.) Take $n \gg 0$ so that it works for all i, j.

$\mathcal{F}(n)$ corresponds to a module M_i' on $D_+(x_i)$, and the map

$$\mathcal{F} \xrightarrow{\cdot x_i^n} \mathcal{F}(n)$$

induces an isomorphism $\begin{array}{c} M_i \longrightarrow M_i' \\ m \longmapsto x_i^n m \end{array}$, since it corresponds to tensoring by a free module of rank 1, and x_i^n is a unit.

Thus, the $x_i^n s_{ij}$ generate M_i' so the t_{ij} generate $\mathcal{F}(n)$ everywhere. $\square$

If X is a projective scheme over A, then by the corollary in the last section, we can write $X = \operatorname{Proj} S$, where

$$S = A[x_0, \dots, x_r]/I$$

and $S_0 = A$, so that I is contained in the irrelevant ideal.

Thus, $A[x_0, \dots, x_r] \longrightarrow S$ induces a closed immersion

$$i : X = \operatorname{Proj} S \to \mathbb{P}^r_A.$$

So we have $f^*(\mathcal{O}(1)) = \mathcal{O}_X(1)$, so the lemma implies

Theorem (Serre) The lemma holds for any projective scheme X over A Noetherian.

Pf: Since $\mathcal{F}$ is coherent and i a closed immersion, for any open affine $U \subseteq \mathbb{P}^r_A$, $i^{-1}(U)$ is also an open affine, so $i_*\mathcal{F}(U)$ is finitely generated, so $i_*\mathcal{F}$ is coherent on $\mathbb{P}^r_A$.

Moreover, $i_*(\mathcal{F}(n)) = i_*\mathcal{F} \otimes \mathcal{O}(n) = (i_*\mathcal{F})(n)$, by the projection formula*.

Thus, $\mathcal{F}(n)$ is globally generated iff $f_*(\mathcal{F}(n))$ is (do you see why?) Thus, we are done by the lemma. $\square$

* The projection formula says that if $f : X \to Y$, $\mathcal{F}$ an $\mathcal{O}_X$-module and $\mathcal{E}$ a locally free $\mathcal{O}_Y$-module, then

$$f_*(\mathcal{F} \otimes f^*\mathcal{E}) \cong f_*\mathcal{F} \otimes \mathcal{E}.$$

(See Har Ex 5.1cd — I might not assign this, but you should think about it regardless!)

This theorem easily implies a useful corollary:

Cor: let X be as in the Theorem, then any coherent sheaf $\mathcal{F}$ is a quotient of a finite direct sum $\bigoplus \mathcal{O}(n)$, where $n \in \mathbb{Z}$.

Pf: We can find m s.t. $\mathcal{F}(m)$ is globally generated by finitely many global sections Thus

$$\bigoplus \mathcal{O}_X \twoheadrightarrow \mathcal{F}(m).$$

Tensoring by $\mathcal{O}_X(-m)$, we get a surjection

$$\bigoplus \mathcal{O}_X(-m) \twoheadrightarrow \mathcal{F}. \quad \square$$

We can also use the theorem to prove an important result about global sections of coherent sheaves, but there is a much easier proof involving cohomology, so we will wait to prove it:

Thm: k a field, A a f.g. k-algebra, X a projective scheme over A, $\mathcal{F}$ coherent. Then $\Gamma(X, \mathcal{F})$ is a finitely generated A module. In particular, if $A = k$, it's a finite-dimensional k-vector space.

Now we can give a condition under which the pushforward of a coherent sheaf is coherent.

Cor: $f: X \to Y$ a projective morphism of schemes of finite type over k. If $\mathcal{F}$ is coherent on X, $f_* X$ is coherent on Y.

Pf: We can check coherence on affine opens, so assume $Y = \operatorname{Spec} A$, where A is a f.g. k-algebra (by finite type).

Since $\mathcal{F}$ is quasi-coherent, $f_* \mathcal{F}$ is at least quasi-coherent, so

$$f_* \mathcal{F} = \widetilde{\Gamma(Y, f_* \mathcal{F})} = \widetilde{\Gamma(X, \mathcal{F})}$$

which is f.g. by the theorem, so $\mathcal{F}$ is coherent. $\square$